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A COMBINATORIAL PRINCIPLE AND ENDOMORPHISM RINGS I: ON p-GROUPS

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Dedicated to the memory of Abraham Robinson on the tenth anniversary of his death

ABSTRACT

Two lines of research are involved here. One is a combinatorial principle, proved in ZFC for many cardinals (e.g., any $\lambda = \lambda^{\aleph_0}$) enabling us to prove things which have been proven using the diamond or for strong limit cardinals of uncountable cofinality. The other direction is building abelian groups with few endomorphisms and/or prescribed rings of endomorphisms. We prove that for a ring R, whose additive group is the p-adic completion of a free p-adic module, $\hat{R}$ is isomorphic to the endomorphism ring of some separable abelian p-group G divided by the ideal of small endomorphisms, with G of power λ for any $\lambda = \lambda^{\aleph_0} \geqq |R|$.

§0. Introduction

Let us first concentrate on the algebraic application, restricting ourselves to separable abelian p-groups [an abelian group is called p-group if $(\forall a \in G)$ $(\exists n > 0)$ $[p^n a = 0]$, and it is called separable if no non-zero element is divisible by p^n for every n].

Such a group cannot be indecomposable (except $\mathbf{Z}/p^n\mathbf{Z}$) because Kulikov proved the existence of many "bounded" direct summands (a direct summand is bounded if a projection on it is small, an endomorphism h of G is small if for every m for every large enough n $[p^n a = 0 \Rightarrow h(p^{n-m}a) = 0]$).

Every p-adic integer gives rise to an endomorphism (if $r = \Sigma\, a_n p^n$, $a_n \in \mathbf{Z}$ and $x \in G$, then we define rx as $\Sigma_{n \leq m}\, a_n p^n x$ for large enough m; we get the same value for every large enough m because G is a p-group).

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So naturally Pierce and Fuchs [3] ask whether there is a separable abelian p-group of power λ which is "essentially indecomposable" or even has only endomorphisms of the form $h_r + h_s$, where h_r is multiplication by a p-adic integer and h_s is small. In [7] this was answered positively for λ strong limit of cofinality $\aleph_0$.[†]

Now Pierce proves more. Let $\mathrm{End}(G)$ be the ring of endomorphisms of G, $E_s(G)$ be the set of small members of $\mathrm{End}(G)$. Pierce proves that

THEOREM. *$E_s(G)$ is an ideal of* $\mathrm{End}(G)$ *and the additive group of* $R(G) \overset{\mathrm{def}}{=} \mathrm{End}(G)/E_s(G)$ *is the p-adic completion of a free p-adic free ring* (see Fuchs [3, vol. I, pp. 193–198]).

Dugas and Gobel succeed in proving a beautiful theorem, the converse of the result mentioned above, thus characterizing the possible $\mathrm{End}(G)/E_s(G)$ (for G a separable abelian p-group). They use the combinatorics of [7], thus inheriting a quite severe restriction on the power of G — it is strong limit of cofinality large enough.

In [7] we mentioned that we can prove the results for more cardinals, so under G.C.H. any $\lambda > \aleph_0$ is O.K.

Here we shall prove the Dugas and Gobel theorem for any cardinality $\lambda = \lambda^{\aleph_0} \geqq |R|$.

Let us now deal with the combinatorial principle. In short, the proof comes from [8] ch. VIII, 1.6, it tries to exhaust the essential part of the proof of [7] §2, and the net result has a flavour of the diamond. In [6] we try to convince that the combinatorial proofs in [8] ch. VIII should be useful generally for proving the existence of many non-isomorphic structures, many, no one embeddable into another and even rigid, indecomposable, etc., structures. As an example we build a rigid Boolean algebra in every uncountable cardinality. As the suggestion has not been followed up, in [9] we develop it for some of the theorems of [8] ch. VIII (with applications to Boolean algebras — on p-groups see [9] p. 106, [11] after Theorem 1); i.e., we develop from it "black boxes" which hopefully can be used by algebraists. Subsequently, the method developed in their paper has been used in some other papers [1, 4, 5].

Another combinatorial idea is embedded into the proof in §2: the p-group consists of countable formal sums of elements of the form $r\eta$ ($r \in R$, $\eta \in {}^{\omega >}\lambda$) (of a specific kind) and is generated by finite sums and a sum of the form

[†] Dugas and Gobel find an error, as we have used a pure closure which is not well defined; but it is not serious. We should just define directly what we used in the cases when we use the pure closure, i.e., in [7, p. 396], defining the a^λ_α: define together $a^\lambda_{\alpha,l}$ $(l < \omega)$, p, $a^\lambda_{\alpha,l+1} - a^\lambda_{\alpha,l} \in G$, $d(a^\lambda_{\alpha,l+1}) =_{\mathrm{ae}} d(a^\lambda_{\alpha,l})$ and replace $\mathrm{PC}(G_\lambda \cup \{a^\kappa_\beta : (\kappa,\beta) < (\lambda,\alpha)\})$ by $\langle G_\lambda \cup \{a^\kappa_{\beta,l} : (\kappa,\beta) < (\lambda,\alpha),\ l < \omega\}\rangle$.

$$\sum_{n \geqq m} p^{n-m}(\eta \restriction n) + \sum_{\eta \in a} r_\eta x_\eta$$

where for some $\zeta < \lambda$, $\eta(n)$ diverge to ζ whereas $\{\eta(l) : l < l(\eta),\ \eta \in a\}$ is a bounded subset of ζ. This gives us much control over the form at the elements.

NOTATION. If $\mathfrak{N}$ is a structure, its universe is M.

We use the usual ordering symbol $\leqq$ for "being an initial segment".

$A \setminus B$ is the set difference and $A \triangle B$ is the symmetric difference.

The author would like to thank Giorgetta for his help. He agreed to referee the paper in handwritten form (thus it has been checked by me quite less than usual), and to totally rewrite the first two sections (namely §1 and §2) amending many things. In particular, in the proof of 2.6 originally we spoke only on the $\eta \in T$ and not $r\eta$'s, forgetting that T generates G as an R-module but not a group. The parts which Giorgetta has not rewritten will appear in a companion paper [10].

§1. Combinatorial tools

Denote by T the tree $({}^{\omega >}\lambda, \leqq)$ of all finite sequences of ordinals smaller than λ. Let L be a set of $\leqq \lambda$ function symbols, each with a finite number of places. Let $\mathfrak{N}$ be the L-algebra freely generated by the set ${}^{\omega >}\lambda$ (so the cardinality of $\mathfrak{N}$ is λ). Fix a strictly increasing continuous mapping $\zeta : \mathrm{cf}\,\lambda \to \lambda$ with $\sup(\mathrm{rg}(\zeta)) = \lambda$. With every node $\eta \in T$ we associate two ordinals $\boldsymbol{l}(\eta)$ and $\boldsymbol{b}(\eta)$ — the length and the breadth of η — as follows:

$$\boldsymbol{l}(\eta) = |\{\theta \in T \mid \theta \leqq \eta\}|, \qquad \boldsymbol{b}(\eta) = \min\{\alpha < \mathrm{cf}\,\lambda \mid \eta \in {}^{\omega >}\zeta(\alpha)\}.$$

Put $T_n = \{\eta \in T \mid \boldsymbol{l}(\eta) = n+1\}$, $T_{>n} = \{\eta \in T \mid \boldsymbol{l}(\eta) \geqq n+2\}$,

$$\boldsymbol{b}(A) = \mathrm{Sup}\{\boldsymbol{b}(\eta) \mid \eta \in A\} \qquad \text{for } A \subseteq T,$$

$$\boldsymbol{b}(a) = \min\{\boldsymbol{b}(A) \mid A \subseteq T,\ a \in \langle A \rangle_{\mathfrak{N}}\} \qquad \text{for } a \in M,$$

$$\boldsymbol{b}(C) = \sup\{\boldsymbol{b}(a) \mid a \in C\} \qquad \text{for } C \subseteq M.$$

Due to the continuity of the mapping ζ the values $\boldsymbol{b}(\eta)$ are successor ordinals for all $\eta \in T$. If $a \in M$ then $a \in \langle A \rangle_{\mathfrak{N}}$ (the substructure of $\mathfrak{N}$ generated by A) for some finite $A \subseteq T$. Thus $\boldsymbol{b}(a)$ is a successor ordinal. For $A \subseteq T$ and $C \subseteq M$ clearly

$$\boldsymbol{b}(A) = \min\{\alpha \mid A \subseteq {}^{\omega >}\zeta(\alpha)\},$$

$$\boldsymbol{b}(C) = \min\{\alpha \mid C \subseteq \langle {}^{\omega >}\zeta(\alpha) \rangle_{\mathfrak{N}}\}.$$

Let $\mathscr{S}$ be the set of all structures $(N, (R_\alpha)_{\alpha<\beta})$ such that

(a) $\beta < \omega_1$,

(b) $N \cap T$ is countable, and there is a countably generated substructure $\mathfrak{M}' \subseteq \mathfrak{M}$ such that $M' = N$,

(c) every R_γ is a countable relation on N.

A plain computation yields $|\mathscr{S}| = \lambda^{\aleph_0}$.

1.1. Definition. Let $(T', \leqq)$ and $(T, \leqq)$ be trees. A mapping $f : T' \to T$ is a *tree embedding* if f is injective, and for all $\eta, \theta \in T'$ we have $l(f(\eta)) = l(\eta)$ and $\eta < \theta \Rightarrow f(\eta) < f(\theta)$.

For $n \leqq \omega$ we denote by $\mathscr{J}_n$ the set of sequences $(f^k, \mathfrak{M}^k)_{k<n}$ with $f^k : {}^{k \geqq}\omega \to {}^{\omega >}\lambda$ tree embedding, $\mathfrak{M}^k \in \mathscr{S}$, $f^{k-1} \subseteq f^k$, $\mathfrak{M}^{k-1} \subseteq \mathfrak{M}^k$ (as a substructure) for every $k < n$, and $\mathrm{rg}(f^k) \subseteq N^k$. $\mathscr{J}$ stands for $\bigcup\{\mathscr{J}_n \mid n \in \omega\}$.

1.2. Definition. A mapping ψ is called a *strategy* if ψ is defined on $\mathscr{J}$ and assigns to every sequence $(f^k, \mathfrak{M}^k)_{k<n}$ from $\mathscr{J}$ a tree embedding $f : {}^{n \geqq}\omega \to {}^{\omega >}\lambda$ with $f^{n-1} \subseteq f$.

We may consider the construction of sequences in $\mathscr{J}_\omega$ as a play where in the n-th move player I chooses f_n and player II chooses $\mathfrak{M}_n$. Then what we call strategies are just strategies for player I. If ψ is a strategy we denote by W'_ψ the set of sequences $(f^n, \mathfrak{M}^n)_{n<\omega}$ such that $(f^n, \mathfrak{M}^n)_{n<j} \in \mathscr{J}_j$ and $f^j = \psi(f^n, \mathfrak{M}^n)_{n<j}$ for all $j \in \omega$. Put

$$W_\psi = \left\{ \left(\bigcup_{n<\omega} f^n, \bigcup_{n<\omega} \mathfrak{M}^n \right) \middle| (f^n, \mathfrak{M}^n)_{n<\omega} \in W'_\psi \right\}.$$

W_ψ can be considered as the set of outcomes of those plays where player I uses the strategy ψ.

1.3. Definition. Let W be a set of pairs $(f, \mathfrak{M})$, where $f : {}^{\omega >}\omega \to {}^{\omega >}\lambda$ is a tree embedding and $\mathfrak{M} \in \mathscr{S}$. ψ is a winning strategy for W if ψ is a strategy, and $W_\psi \subseteq W$. W is a *barrier* if there exists a winning strategy for W. W is a *disjoint barrier* if W is a barrier, and for distinct $(f, \mathfrak{M}), (f', \mathfrak{M}') \in W$ there is no common branch to the trees $\mathrm{rg}(f)$ and $\mathrm{rg}(f')$.

1.4. Theorem. *Suppose* $\mathrm{cf}\,\lambda > \omega$. *Then there is an ordinal* $\alpha^* < \lambda^+$ *and a family* $W = \{(f_\alpha, \mathfrak{M}_\alpha) \mid \alpha < \alpha^*\}$ *with the following properties*:

(a) *W is a disjoint barrier.*

(b) $\mathrm{cf}\,\boldsymbol{b}(N_\alpha) = \omega$ *for all* $\alpha < \alpha^*$.

(c) $\boldsymbol{b}(N_\alpha) \leqq \boldsymbol{b}(N_\beta)$ *whenever* $\alpha < \beta < \alpha^*$.

(d) $\boldsymbol{b}(v) = \boldsymbol{b}(N_\alpha)$ *for all* $\alpha < \alpha^*$ *and every branch* $v \subseteq \mathrm{rg}(f_\alpha)$.

(e) *If* $\beta + 2^{\aleph_0} \leqq \alpha < \alpha^*$ *then* $v \cap N_\beta$ *is finite for every branch* $v \subseteq \mathrm{rg}(f_\alpha)$. (*Here* $\beta + 2^{\aleph_0}$ *is the ordinal sum.*)

Proof. We confine ourselves to the special case $\lambda^{\aleph_0} = \lambda$ since this condition will be satisfied in our application. A slight modification of the argument yields a proof for the general case cf $\lambda > \omega$. (Replace the enumeration below of $\mathcal{J}$ in λ by a 1-1 mapping of $\mathcal{J}$ into the set of branches of ${}^{\omega>}\lambda$. Modify the function φ using the fact that distinct branches already differ on finite levels.)

(a) We infer $|\mathcal{J}| = \lambda$ from $|\mathcal{J}| = |\mathcal{S}|$ and the assumption $\lambda = \lambda^{\aleph_0}$. So we can fix an enumeration $(g_\gamma)_{\gamma<\lambda}$ of $\mathcal{J}$. For $g = (f_k, \mathfrak{M}_k)_{k<n} \in \mathcal{J}$ denote by $\boldsymbol{b}(g)$ the ordinal $\max\{\boldsymbol{b}(N_k) \mid k < n\}$.

Let $\varphi : \lambda \times \lambda \times \omega \to \lambda$ be a 1-1 function such that $\varphi(\alpha, \beta, n) \geqq \beta$ for all $(\alpha, \beta, n) \in \lambda \times \lambda \times \omega$. We define a strategy ψ by induction. $\psi \restriction \mathcal{J}_0$ is defined by $\psi(\emptyset) = \{(\emptyset, \emptyset)\}$.

For $g \in \mathcal{J}_{n+1}$, say $g = (f^k, \mathfrak{M}^k)_{k<n}$, $g = g_\gamma$ in the enumeration of $\mathcal{J}$, we define $\psi(g)$ to be the mapping $f : {}^{n\geq}\omega \to {}^{\omega>}\lambda$ which extends f^{n-1} and satisfies

$$f(\eta^\frown\langle j\rangle) = f^{n-1}(\eta)^\frown\langle\varphi(\gamma, \boldsymbol{b}(g), j)\rangle \qquad \text{for all } \eta \in {}^{n-1}\omega \quad \text{and} \quad j \in \omega.$$

(The sign $^\frown$ denotes concatenation of sequences.)

Using the injectivity of φ in the third argument one sees that f is a tree embedding, and so ψ is a strategy and W_ψ is a barrier.

To establish the disjointness of W_ψ take elements $(f_1, \mathfrak{M}_1)$ and $(f_2, \mathfrak{M}_2)$ of W_ψ, and assume that the ranges of f and f' contain a common branch v.

$(f_i, \mathfrak{M}_i)$ $(i = 1, 2)$ can be represented as

$$(f_i, \mathfrak{M}_i) = \left(\bigcup_{n<\omega} f_i^n, \bigcup_{n<\omega} \mathfrak{M}_i^n \right)$$

with

$$f_i^n = \psi(f_i^k, \mathfrak{M}_i^k)_{k<n} \quad \text{and} \quad (f_i^k, \mathfrak{M}_i^k)_{k<n} = g_{\gamma(i,n)}.$$

Since the f_i^n are tree embeddings we find for $\theta \in T_n \cap v$ sequences $\eta, \eta' \in {}^n\omega$ with $f_1^n(\eta) = f_2^n(\eta') = \theta$.

Comparing the last components of $f_1^n(\eta)$ and $f_2^n(\eta')$ we get

$$\varphi(\gamma(1, n), \boldsymbol{b}(g_{\gamma(1,n)}), \eta(n-1)) = \varphi(\gamma(2, n), \boldsymbol{b}(g_{\gamma(2,n)}), \eta'(n-1)).$$

Since φ is 1-1 we conclude that $\gamma(1, n) = \gamma(2, n)$ for all $n < \omega$. Hence $(f_1, \mathfrak{M}_1) = (f_2, \mathfrak{M}_2)$, and so W_ψ is a disjoint barrier.

(b), (d) Let $(f, \mathfrak{M}) \in W_\psi$. $(f, \mathfrak{M})$ can be represented by

$$(f,\mathfrak{M})=\left(\bigcup_{n<\omega} f^n,\ \bigcup_{n<\omega}\mathfrak{M}^n\right),\quad (f^k,\mathfrak{M}^k)_{k<n}=g_{\gamma(n)},\quad f^n=\psi(g_{\gamma(n)}).$$

Let v be a branch of $\mathrm{rg}(f)$, say $v\cap T_{n-1}=\{\theta_n\}$. As f is a tree embedding we find a branch $w\subseteq T$ with $w\cap T_{n-1}=\{\eta_n\}$ and $f(\eta_n)=f^n(\eta_n)=\theta_n$. Remembering the definition of φ and using $\boldsymbol{b}(\boldsymbol{b}(A))\geqq \boldsymbol{b}(A)+1$ for all $A\subseteq M$ we establish the following chain of inequalities:

$$\boldsymbol{b}(N^n)\geqq \boldsymbol{b}(\mathrm{rg}(f^n))\geqq \boldsymbol{b}(\theta_n)=\boldsymbol{b}(f(\eta_{n-1})^\wedge\varphi(\gamma(n),\boldsymbol{b}(g_\gamma(n)),\eta(n-1)))$$

$$\geqq \boldsymbol{b}(\varphi(\gamma(n),\boldsymbol{b}(g_{\gamma(n)}),\eta(n-1))\geqq \boldsymbol{b}(\boldsymbol{b}(g_{\gamma(n)}))>\boldsymbol{b}(g_{\gamma(n)})\geqq \boldsymbol{b}(N^{n-1}).$$

So $\boldsymbol{b}(N^{n-1})<\boldsymbol{b}(v\cap N^n)\leqq \boldsymbol{b}(N^n)$. This implies $\mathrm{cf}\,\boldsymbol{b}(N)=\omega$, and $\boldsymbol{b}(N)=\boldsymbol{b}(v)$ for all branches v of $\mathrm{rg}(f)$.

(c) and (e). Start with an arbitrary enumeration $(f'_\gamma,\mathfrak{M}'_\gamma)_{\gamma<\lambda}$ of W_ψ. For $(f,\mathfrak{M})\in W_\psi$ put

$$\mathrm{Nb}(f,\mathfrak{M})=\{(f',\mathfrak{M}')\in W_\psi \mid \text{there is a branch } v\subseteq \mathrm{rg}(f') \text{ such that } |v\cap N|=\aleph_0\}.$$

Put $\mathrm{Nb}_1(f,\mathfrak{M})=\mathrm{Nb}(f,\mathfrak{M})$, $\mathrm{Nb}_{n+1}(f,\mathfrak{M})=\bigcup\{\mathrm{Nb}(f',\mathfrak{M}')\mid (f',\mathfrak{M}')\in \mathrm{Nb}_n(f,\mathfrak{M})\}$ and $\mathrm{Nb}_\omega(f,\mathfrak{M})=\bigcup\{\mathrm{Nb}_n(f,\mathfrak{M})\mid n<\omega\}$.

By the disjointness of W_ψ we get $|\mathrm{Nb}(f,\mathfrak{M})|\leqq|\{v\mid v \text{ branch of } T,\ |v\cap N|=\aleph_0\}|$. So condition (b) of the definition of φ yields $|\mathrm{Nb}(f,\mathfrak{M})|\leqq 2^{\aleph_0}$. Hence $|\mathrm{Nb}_\omega(f,\mathfrak{M})|\leqq 2^{\aleph_0}$.

We partition the set $U_\alpha=\{(f,\mathfrak{M})\in W_\psi \mid \boldsymbol{b}(N)=\alpha\}$ ($\alpha<\mathrm{cf}\,\lambda$) into classes U_α^β ($\beta<\mu_\alpha,\ \mu_\alpha\leqq\lambda$) as follows:

$$U_\alpha^\beta=\mathrm{Nb}_\omega(f'_\gamma,\mathfrak{M}'_\gamma)\setminus\bigcup_{\delta<\beta}U_\alpha^\delta,$$

where γ is the first ordinal with $(f'_\gamma,\mathfrak{M}'_\gamma)\notin\bigcup_{\delta<\beta}U_\alpha^\delta$ and $\boldsymbol{b}(N'_\gamma)=\alpha$.

Every class U_α^β can be equipped with a wellordering $<_\alpha^\beta$ of type $\leqq 2^{\aleph_0}$. Now define a wellordering $<$ on W_ψ as follows: $(f,\mathfrak{M})<(f',\mathfrak{M}')$ if

either $\boldsymbol{b}(N)<\boldsymbol{b}(N')$

or $\boldsymbol{b}(N)=\boldsymbol{b}(N')=\alpha$, and $(f,\mathfrak{M})\in U_\alpha^\beta$, and $(f',\mathfrak{M}')\in U_\alpha^{\beta'}$ for some $\beta<\beta'$

or $\boldsymbol{b}(N)=\boldsymbol{b}(N')=\alpha$, and $(f,\mathfrak{M}),\ (f',\mathfrak{M}')\in U_\alpha^\beta$ and $(f,\mathfrak{M})<_\alpha^\beta(f',\mathfrak{M}')$.

The wellordering $<$ induces an enumeration $(f_\alpha,\mathfrak{M}_\alpha)_{\alpha<\alpha^*}$ ($\alpha^*<\lambda^+$) which clearly satisfies the conclusions (c) and (e) of the theorem.

§2. Endomorphism rings of abelian p-groups

A homomorphism $h:G\to H$ between abelian p-groups is *small* if for every $s<\omega$ there exists $n<\omega$ such that for all $a\in G$ if $p^na=0$ then $p^{n-s}h(a)=0$.

The small endomorphisms of an abelian group G constitute an ideal $E_s(G)$ of the ring $\mathrm{End}(G)$ of all endomorphisms of G.

In the sequel we say group for abelian group, and homomorphism (endomorphism) for group homomorphism (endomorphism). So for a module G we denote by $\mathrm{End}(G)$ the ring of group endomorphisms (and not of module endomorphisms) of G.

For G an R-module and $r \in R$ we use h_r to design the endomorphism $G \to G$, $a \to ra$.

In the sequel the notation h_r appears in combinations like "$h - h_r$", where h is an endomorphism. Then it is understood that h_r has the same domain as h.

The whole section is devoted to the proof of the following

2.1. THEOREM. *Let R be a ring whose additive group is the p-adic completion of a free p-adic module. Then for every cardinal λ with $\lambda = \lambda^{\aleph_0}$ and $\lambda \geqq |R|$ there is a family $(G_\alpha)_{\alpha<2^\lambda}$ of separable abelian p-groups with the following properties*:

(1) *Each G_α is also an R-module and $|G_\alpha| = \lambda$ for all $\alpha < 2^\lambda$.*

(2) *If $h : G_\alpha \to G_\beta$ is a nonsmall homomorphism then $\alpha = \beta$, and there exists $r \in R$ such that $h - h_r$ is small.*

(3) $\mathrm{End}(G_\alpha) = R \oplus E_s(G_\alpha)$ *is a split extension for all $\alpha < 2^\lambda$.* (For a definition of split extension see [2, p. 360].)

PROOF. Let R be as in the theorem. For our purposes the following properties of R are needed: The additive group R^+ of R is torsion free, complete in the p-adic topology and reduced, i.e., R^+ contains no sequence $(a_n)_{n<\omega}$ such that $pa_{n+1} = a_n \neq 0$ for all $n < \omega$.

Let G be the R-module $\bigoplus_{\eta\in T}(R\cdot\eta/p^{l(\eta)+1}R\cdot\eta)$ where T is as in §1 and $R\cdot\eta$ is the cyclic R-module freely generated by $\{\eta\}$. By a suitable choice of coset representatives we get $T \subseteq G$, and $G = \langle T\rangle_G$. Denote by $\hat{G}$ the p-adic torsion completion of G. $\hat{G}$ carries a natural R-module structure. Every element of $\hat{G}$ can be represented by a formal sum $\Sigma_{\eta\in A}\, r_\eta\cdot\eta$ with $A \subseteq T$ countable and $r_\eta \in R$ are such that $r_\eta\cdot\eta$ are nonzero elements of G of bounded order and for every n, $A \cap T_n$ is finite. Conversely each such sum represents an element of $\hat{G}$, and the componentwise sum taken in G of two formal sums represents the sum of the corresponding elements of $\hat{G}$. While the representation of an element $a \in \hat{G}$ by a sum $\Sigma_{\eta\in A}\, r_\eta\cdot\eta$ is not unique, the index set A is uniquely determined. We call this set the *support* of a and denote it by $\mathrm{supp}(a)$.

For $n < \omega$ and a as above we define an element $a_{(n)} \in G$ by $a_{(n)} = \Sigma_{\eta\in A\cap T_{\leq n}}\, r_\eta\cdot\eta$.

In the sequel G takes the role of $\mathfrak{N}$ in section 1, identifying formal sums with the set of their summands, the (representations of) elements of $\hat{G}$ become subsets of G, and so we have $l(a)=l(\mathrm{supp}(a))$ and $b(a)=b(\mathrm{supp}(a))$ for $a\in\hat{G}$.

For every $a\in\hat{G}$ there exists a minimal nonnegative integer s such that a has the representation

$$a=\sum_{\eta\in A} r_\eta (p^{l(\eta)-s})\eta.$$

Here we allow the integer $l(\eta)-s$ to be negative.

Using this representation for a we define for $m\in\omega$:

$$a^m=\sum_{\substack{\eta\in A\\ l(\eta)\geq s+m}} r_\eta (p^{l(\eta)-s-m})\eta.$$

For $A\subseteq T$ countable, $A\cap T_n$ finite for every n, we denote by a_A the element of $\hat{G}$ represented by $\sum_{\eta\in A} p^{l(\eta)}\eta$. So the order of a_A is p, and

$$a_A^m=\sum_{\substack{\eta\in A\\ l(\eta)\geq m}} (p^{l(\eta)-m})\eta.$$

Instead of $a^m_{\{\eta\}}$ we shall write a^m_η.

Now we are going to apply Theorem 1.4 with $G=\mathfrak{N}$. We want to define by induction on α branches v_α of $\mathrm{rg}(f_\alpha)$ and elements $a_\alpha\in\hat{N}_\alpha$ (the p-adic completion of N_α), as well as a subset $J\subseteq\alpha^*$ and elements $b_\alpha\in\hat{N}_\alpha$ for $\alpha\in J$. For $J'\subseteq a^*$ we put $G(J')=SG(G\cup\{a^m_\alpha \mid \alpha\in J',\ m\in\omega\})$. (For $H\subseteq\hat{G}$, the R-submodule of $\hat{G}$ generated by H is designated by $SG(H)$.) The aim of this definition is to produce groups $G(J')$ which, for a suitably chosen system of subsets $J'\subseteq\alpha^*$, satisfy the conditions of Theorem 2.1.

2.2. Definition. We define v_α, a_α for $\alpha<\alpha^*$, J (i.e., the truth value of $\alpha\in J$), b_α for $\alpha\in J$: Let $\alpha\in J$ iff $\alpha<\alpha^*$ and the following conditions are satisfied.

(i) There exist $c\in\hat{N}_\alpha$ and $h\in\mathrm{End}(\hat{G})$ such that $\mathfrak{M}_\alpha=(N_\alpha,R_\alpha(h,c),c)$ where $R_\alpha(h,c)$ denotes the relation $\{(a,(ha)_{(n)}) \mid a\in\mathrm{rg}(f_\alpha)\cup c,\ n<\omega\}$, $b(c)<b(N_\alpha)$, $b(h(c))<b(N_\alpha)$. (So h maps $\mathrm{rg}(f_\alpha)\cup c$ into N_α as $\mathrm{rg}(f_\alpha)\subseteq N_\alpha$.)

(ii) Either

(a) There exists a branch $v\subseteq\mathrm{rg}(f_\alpha)$ such that for $a=a_v+c$ and $\gamma\in J\cap\alpha$ the following holds:

$$(*)\qquad h(a),\ b_\gamma\notin SG(G(\alpha)\cup\{a^s \mid s\in\omega\}),$$

or

(b) There is no branch v as in (a) but there is a branch $v \subseteq \mathrm{rg}(f_\alpha)$ such that for all $\gamma \in J \cap \alpha$ condition (*) holds for a_v instead of a.

In case (ii)(a) let v_α be an arbitrary branch of $\mathrm{rg}(f_\alpha)$ satisfying (*), and put $a_\alpha = a_{v_\alpha} + c$. In case (ii)(b) proceed in the same way for a_{v_α} instead of $a_{v_\alpha} + c$. In either case put $b_\alpha = h(a_\alpha)$ for some $h \in \mathrm{End}(\hat{G})$ with $\mathfrak{M}_\alpha = (N_\alpha, R_\alpha(h, c), c)$. Clearly the value $h(a_\alpha)$ is independent of the particular choice of h. If $\alpha < \alpha^*$, $\alpha \notin J$ let v_α be a branch of $\mathrm{rg}(f_\alpha)$ such that $b_\gamma \notin SG(G(\alpha) \cup \{a^s_{v_\alpha} \mid s < \omega\})$, for all $\gamma \in J \cap \alpha$, and put $a_\alpha = a_{v_\alpha}$. That such a branch exists is seen as follows: Assume that $\gamma \in J \cap \alpha$ and $b_\gamma \in SG(G(\alpha) \cup \{a^s_v \mid s < \omega\})$ for a branch $v \subseteq \mathrm{rg}(f_\alpha)$. Then $\mathrm{supp}(b_\gamma) \cap v$ is infinite since else $b_\gamma \in G(\alpha)$, contradicting the induction hypothesis. As $b_\gamma \in \hat{N}_\gamma$ we get $\gamma + 2^{\aleph_0} > \alpha$ as a consequence of Theorem 1.4e. Since $|\{\gamma \mid \gamma < \alpha < \gamma + 2^{\aleph_0}\}| < 2^{\aleph_0}$ and $\mathrm{rg}(f_\alpha)$ contains $2^{\aleph_0}$ branches we can find a branch $v_\alpha \subseteq \mathrm{rg} f_\alpha$ such that $\mathrm{supp}(b_\gamma) \cap v_\alpha$ is finite for all $\gamma \in J \cap \alpha$ with $\alpha < \gamma + 2^{\aleph_0}$, and thus (by a second application of Theorem 1.4(e)) for all $\gamma \in J \cap \alpha$.

NOTATION. $A \subseteq^* B$ means $a \in B$ for all but finitely many $a \in A$, "$A =^* B$" stands for "$A \subseteq^* B$ and $B \subseteq^* A$".

2.3. PROPOSITION. (1) $|\alpha^* \setminus J| = \lambda$.

(2) *$b_\gamma \notin G(\alpha^*)$ for all $\gamma \in J$.*

(3) *Every element a of $G(\alpha^*)$ admits the following representations for every sufficiently large integer s:*

First standard representation:

$$a = \sum_{1 \leqq i \leqq k_1} r_i a^s_{\alpha_i} + c \qquad \text{with } 0 \leqq k_1 < \omega, \quad r_i \in R \quad \text{and}$$

$$r_i a^s_{\alpha_i} \neq 0 \qquad \text{for } 1 \leqq i \leqq k_1, \quad \alpha_1 > \cdots > \alpha_{k_1}, \quad c \in G.$$

Second standard representation:

$$a = \sum_{1 \leqq i \leqq k} r'_i a^s_{v_{\alpha_i}} + b + c' \qquad \text{with } 0 \leqq k < \omega, \quad r'_i \in R \quad \text{and}$$

$$r'_i a^s_{v_{\alpha_i}} \neq 0 \qquad \text{for } 1 \leqq i \leqq k, \quad \alpha_1 > \cdots > \alpha_k, \quad b \in \hat{G}, \quad c' \in G,$$

$$\boldsymbol{b}(b) < \boldsymbol{b}(v_{\alpha_1}) = \cdots = \boldsymbol{b}(v_{\alpha_k}) < \boldsymbol{b}(\theta) \qquad \text{for every } \theta \in c'.$$

The numbers k_1, k and α_i ($1 \leqq i \leqq k_1$) are uniquely determined and the same for all s. k is the number of subscripts α_i with $\boldsymbol{b}(v_{\alpha_i}) = \boldsymbol{b}(v_{\alpha_1})$ in the first *standard representation. The ordinals* $\alpha_1, \dots, \alpha_k$ *are the same in both representations. The elements* c, b, c' *are uniquely determined for every* s. *Moreover, we have* $k > 0$ *for* $a \notin G$ *and* $v_{\alpha_i} \subseteq^* \operatorname{supp}(a)$ *for* $1 \leqq i \leqq k$.

(4) *For every* $a \in G(\alpha^*) \setminus G$ *there is* $\alpha < \alpha^*$ *such that* $a \in G(\alpha + 1) \setminus G(\alpha)$. *Moreover* $\boldsymbol{b}(a) = \boldsymbol{b}(N_\alpha)$.

(5) $G(J')$ *is a pure subgroup of* $\hat{G}$ *for every* $J' \subseteq \alpha^*$.

(6) *For* $J' \subseteq \alpha^*$, *every homomorphism from* $G(J')$ *to* $\hat{G}$ *extends to an endomorphism of* $\hat{G}$.

PROOF. (1), (2), (3), (4) are simple consequences of the definitions. The fact is used that the family $(\mathfrak{M}_\alpha, f_\alpha)_{\alpha < \alpha^*}$ is a disjoint barrier. In the second standard representation, $\boldsymbol{b}(v_{\alpha_i}) < \boldsymbol{b}(\theta)$ for all $\theta \in c$ can be obtained since $\boldsymbol{b}(v_{\alpha_i})$ is a limit ordinal, and so $\boldsymbol{b}(v_{\alpha_i}) \neq \boldsymbol{b}(\eta)$ for all $\eta \in T$.

(5) Let $a \in G(J^*)$, $n \in \omega$. Using the first standard representation we get

$$a = \sum_{1 \leqq i \leqq k} r_i a_{\alpha_i}^{\backslash} + c = \sum_{1 \leqq i \leqq k} p^n r_i a_{\alpha_i}^{n+\backslash} + c' \qquad \text{for some } c' \in G.$$

Since G is a pure subgroup of $\hat{G}$ and a subgroup of $G(J')$, the element a is divisible by p^n in $\hat{G}$ iff c' is divisible by p^n in $\hat{G}$ iff it is divisible by p^n in $G(J')$ iff a is divisible by p^n in $G(J')$.

(6) As $G(J')$ is pure in $\hat{G}$ and $\hat{G}$ is torsion complete, this follows from the implication (i)→(iii) of theorem 68.4 in [3].

2.4. DEFINITION. An endomorphism h of $\hat{G}$ is *almost constant* if there is a small endomorphism h' of $\hat{G}$ and an element $r \in R$ with $h = h' + h_r$.

Theorem 2.1 reduces to the following lemma:

2.5. LEMMA. (a) *Every endomorphism of* $\hat{G}$ *which maps* $G(J)$ *into* $G(\alpha^*)$ *is almost constant.*

(b) *If* J_1 *and* J_2 *are subsets of* α^* *containing* J *such that* $J_1 \not\subseteq J_2$ *then all homomorphisms from* $G(J_1)$ *to* $G(J_2)$ *are small.*

By Proposition 2.3(1) it is easy to find a collection $\{J_\alpha \mid \alpha < 2^\lambda\}$ of subsets of α^* containing J such that $J_\alpha \not\subseteq J_\beta$ for $\alpha \neq \beta$. By Proposition 2.3(6) for $\alpha < 2^\lambda$ every endomorphism of $G(J_\alpha)$ extends to an endomorphism of $\hat{G}$ which maps $G(J)$ into $G(\alpha^*)$. So putting $G_\alpha = G(J_\alpha)$ and using Lemma 2.5, Theorem 2.1 is established. (Part 3 of the theorem follows easily from part 2 and the choice of G.)

PROOF OF LEMMA 2.5. (b) Let h be a homomorphism from $G(J_1)$ to $G(J_2)$.

Assume for contradiction that h is nonsmall. By Proposition 2.3(6) h extends to an endomorphism $\bar{h}$ of $\hat{G}$. Clearly $\bar{h}$ is nonsmall and maps $G(J)$ into $G(\alpha^*)$. So by (a) $\bar{h}$ is almost constant. Say $\bar{h} = h' + h_r$ with $h' \in E_s(\hat{G})$ and $h_r \neq 0$. Pick $\alpha \in J_1 \setminus J_2$. For a suitable integer m_0 there is a positive integer, say j, such that the order of a_α^m is p^{m+j} for all $m \geqq m_0$. As R is reduced we can choose $m \geqq m_0$ such that $r \not\equiv 0 \bmod p^m$. Since h' is small we find an integer n such that $h' p^{n+j} a_\alpha^{n+m} = 0$. (Indeed $p^{n+j} a_\alpha^{n+m} = p^{n-m} p^{m+j} a_\alpha^{n+m}$, and $p^n p^{m+j} a_\alpha^{n+m} = 0$ by the choice of j.)

Put $p^{n+j} a_\alpha^{n+m} = a'$. So $ha' = ra'$. As $a' \in G(J_1)$ and $hG(J_1) \subseteq G(J_2)$ we get $ra' \in G(J_2)$. Thus we have the representation

$$ra' = \sum_{1 \leqq i \leqq k} r_i a_{\alpha_i}^s + c \qquad \text{for some } s, k \in \omega,$$

$\alpha_1, \ldots, \alpha_k \in J_2$, $r_1, \ldots, r_k \in R$ and $c \in G$. The integers j and m were chosen so that ra' has an infinite support. Consequently $k \geqq 1$. But $\alpha \notin J_2$, contradicting the uniqueness of the ordinals α_i in the first standard representation.

(a) For this part we use a further lemma whose proof is postponed.

2.6. Lemma. *If $h \in \operatorname{End}(\hat{G})$ is not almost constant then there is $c \in \hat{G}$ with $h(c) \notin SG(G(\alpha^*) \cup \{c^s \mid s < \omega\})$.*

Now assume that h is an endomorphism of $\hat{G}$ which is not almost constant. We are going to find $\alpha \in J$ and $c \in \hat{G}$ such that $\mathfrak{M}_\alpha = (N_\alpha, R_\alpha(h, c), c)$. Then $h(a_\alpha) = b_\alpha \notin G(\alpha^*)$, and so $hG(J) \not\subseteq G(\alpha^*)$, which proves (a). Choose an element c as in Lemma 2.6. Since $(\mathfrak{M}_\alpha, f_\alpha)_{\alpha < \alpha^*}$ is a barrier we find $\beta < \alpha^*$ such that $\mathfrak{M}_\beta = (N_\beta, R_\beta(h, c), c)$, and $\boldsymbol{b}(c)$, $\boldsymbol{b}(hc) < \boldsymbol{b}(N_\beta)$. (Play for player II.) We want to show $\beta \in J$.

As we have seen, defining a_α for $\alpha \in \alpha^* \setminus J$ there exists a branch $v \subseteq \operatorname{rg}(f_\beta)$ such that $b_\gamma \notin SG(G(\beta) \cup \{a_v^s \mid s < \omega\})$ for all $\gamma \in J \cap \beta$. It is readily seen that also $b_\gamma \notin SG(G(\beta) \cup \{(a_v + c)^s \mid s < \omega\})$ for all $\gamma \in J \cap \beta$, since $\boldsymbol{b}(c) < \boldsymbol{b}(v)$. So it remains to be shown that $h(a) \notin SG(G(\beta) \cup \{a^s \mid s < \omega\})$ for some $a \in \{a_v, a_v + c\}$.

Assume for contradiction that

$$h(a_v + c) - r(a_v + c)^s \in G(\beta) \quad \text{and} \quad ha_v - r' a_v^{s'} \in G(\beta).$$

Subtraction yields

$$h(c) + r' a_v^{s'} - r a_v^s - r c^s \in G(\beta).$$

As $\boldsymbol{b}(c)$, $\boldsymbol{b}(hc) < \boldsymbol{b}(v)$, and $v \cap \operatorname{supp}(a)$ is finite for every $a \in G(\beta)$, the set $\operatorname{supp}(r' a_v^{s'} - r a_v^s)$ must be finite. Since nonzero elements of R are not divisible by infinitely many powers of p one easily concludes that $ra^s_\nu - r'a^{s'}_\nu = 0$. So $h(c) - rc^s \in G(\beta)$, which contradicts the initial assumption that $h(c) \notin SG(G(\alpha^*) \cup \{c^s \mid s \in \omega\})$.

Proof of Lemma 2.6. Let h satisfy the conditions of the Lemma.

Case 1

$$(1)\quad \begin{cases} \text{For every } s \in \omega \text{ there exists } n \in \omega \text{ such that for all } \eta \in T \text{ with} \\ l(\eta) \geqq n \text{ and for all } r \in R \text{ the set } \operatorname{supp}(h(ra^s_\eta)) \text{ is contained in } \{\eta\}. \end{cases}$$

For the following case distinction we consider the set $\mathscr{C}$ of all subsets C of $R \times T$ satisfying the following conditions:

(i) If $(r, \eta), (r', \eta) \in C$ then $r = r'$.

(ii) For every $n \in \omega$ there is $\eta \in T$ such that $l(\eta) \geqq n$ and $(1, \eta) \in C$.

(iii) The set $\{(r, \eta) \in C \mid l(\eta) \leqq n\}$ is finite for all $n < \omega$.

(iv) $\omega \setminus \{l(\eta) \mid (r, \eta) \in C$ for some $r \in R\}$ is infinite

Subcase A

$$(2)\quad \begin{cases} s \in \omega \text{ and } C \in \mathscr{C} \text{ can be chosen such that for all } r \in R \text{ and} \\ m, n < \omega \text{ there is a pair } (r', \eta) \in C \text{ with } l(\eta) \geqq n \text{ and} \\ h(r'a^s_\eta) \neq rr'a^{s+m}_\eta. \end{cases}$$

Pick $s \in \omega$ and $C \in \mathscr{C}$ suitable for (2). Put

$$c = \sum_{\substack{(r', \eta) \in C \\ l(\eta) \geqq n_s}} r'a^s_\eta$$

where n_s according to (1) is chosen to satisfy $\operatorname{supp}(h(r'a^s_\eta)) \subseteq \{\eta\}$ for all η with $l(\eta) \geqq n_s$.

It follows from condition (iii) that $c \in \hat{G}$. The continuity of h in the p-adic topology ensures that

$$h(c) = \sum_{\substack{(r', \eta) \in C \\ l(\eta) \geqq n_s}} h(r'a^s_\eta).$$

Choose $r \in R$, $m \in \omega$.

(1) yields $\operatorname{supp}(h(c) - rc^m) \subseteq \{\eta \mid (r', \eta) \in C$ for some $r' \in R\}$. So we infer from condition (iv) that the set $\omega \setminus \{l(\eta) \mid \eta \in \operatorname{supp}(h(c) - rc^m)\}$ is infinite. Consequently $h(c) - rc^m \notin G(\alpha^*) \setminus G$, as, by Proposition 2.3(3),

$$|\omega \setminus \{l(\eta) \mid \eta \in \operatorname{supp}(a)\}| < \aleph_0 \qquad \text{for } a \in G(\alpha^*) \setminus G.$$

An application of the conditions (1) and (i) gives the result

$$\operatorname{supp}(h(c)-rc^m)\supseteq \bigcup\{\operatorname{supp}(h(r'a^s_\eta)-rr'a^{s+m}_\eta)\mid (r',\eta)\in C, l(\eta)\geqq n_s+m\}.$$

Using (2) and (i) we conclude that the set $\{l(\eta)\mid \eta\in\operatorname{supp}(h(c)-rc^m)\}$ is infinite. So $h(c)-rc^m\notin G$ and we have shown that $h(c)\notin SG(G(\alpha^*)\cup\{c^m \mid m\in\omega\})$.

Subcase B

(3) $\begin{cases}\text{For all } s\in\omega \text{ and } C\in\mathscr{C} \text{ there are } r\in R,\ m\in\omega \text{ and } n\in\omega\\ \text{such that for every pair } (r',\eta)\in C \text{ with } l(\eta)\geqq n \text{ the equation}\\ h(r'a^s_\eta)=rr'a^{s+m}_\eta \text{ holds.}\end{cases}$

Instead of the equation in (3) we can write $h(r'a^s_\eta)=\bar r a^s_\eta$ for some $\bar r\in R$ since h is a homomorphism, and $\operatorname{supp}(h(r'a_\eta))\subseteq\{\eta\}$. If one compares the two equations using the properties of R one sees that p^m divides rr'. So by condition (ii) in the definition of $\mathscr{C}$, also r is divisible by p^m. Thus we can assume $m=0$ in the equation in (3).

For $s<\omega$ and $C\in\mathscr{C}$ take $r(s,C)\in R$ and $n(s,C)\in\omega$ appropriate for (3). By condition (ii) in the definition of $\mathscr{C}$ we can choose $\eta\in T$ such that $(1,\eta)\in C$ and $l(\eta)\geqq\max(n(s,C),n(s+1,C))$. Using (3) for $(1,\eta)$ we get $h(a^s_\eta)=r(s,C)a^s_\eta$ and $h(a^{s+1}_\eta)=r(s+1,C)a^{s+1}_\eta$. Since $a^s_\eta=pa^{s+1}_\eta$ it follows that $r(s,C)\equiv r(s+1,C)\bmod p^{s+1}$ for all $s<\omega$. Thus we find elements $t_k(C)$ of R such that $r(s,C)=\Sigma_{k\leqq s}t_k(C)\cdot p^k$.

Put $r(C)=\Sigma_{k<\omega}t_k(C)\cdot p^k$. Since R is complete in the p-adic topology the element $r(C)$ is contained in R, and we have $r(C)r'a^s_\eta=r(s,C)r'a^s_\eta$ for all $r'\in R$, $\eta\in T$ and $s<\omega$. Next we show that r can be chosen independent of C, too. Take B, $C\in\mathscr{C}$. We easily find $D\in\mathscr{C}$ with $B\cap D\in\mathscr{C}$ and $C\cap D\in\mathscr{C}$. For every s we find $(1,\eta)\in B\cap D$ such that $r(B)a^s_\eta=r(D)a^s_\eta$. Consequently $r(B)\equiv r(D)$ mod p^{s+1} for all $s<\omega$.

Since R has no elements of infinite height it follows that $r(B)=r(D)$, and in the same way we get $r(C)=r(D)$. So $r(B)=r(C)$.

Up to now we have improved (3) to the following statement:

(4) $\begin{cases}\text{There is } r\in R \text{ with the following property:}\\ \text{For all } s\in\omega \text{ and } C\in\mathscr{C} \text{ there exists } n\in\omega \text{ such that}\\ \text{for all } (r',\eta)\in C \text{ with } l(\eta)\geqq n \text{ the equation } h(r'a^s_\eta)=rr'a^s_\eta \text{ holds.}\end{cases}$

We want to deduce an even stronger statement, namely:

(5) $\begin{cases}\text{There is } r\in R \text{ with the following property:}\\ \text{For all } s\in\omega \text{ there exists } n\in\omega \text{ such that for all } C\in\mathscr{C}\\ \text{and all } (r',\eta)\in C \text{ with } l(\eta)\geqq n \text{ the equation } h(r'a^s_\eta)=rr'a^s_\eta \text{ holds.}\end{cases}$

Statement (5) tells us that h is almost constant. So one reaches a contradiction with the assumption of Lemma 2.6, and subcase B is shown to be impossible.

Statement (5) is verified indirectly. Take $r \in R$ appropriate for (4). Choose for this r an integer s and for every $n \in \omega$ a set $C_n \in \mathscr{C}$ and a pair $(r'_n, \eta_n) \in C_n$ such that $l(\eta_n) \geqq n$ and $h(r_n a^s_{\eta_n}) \neq r r'_n a^s_{\eta_n}$.

This can be done assuming the negation of (5). It is easy to find $C \in \mathscr{C}$ such that the set $\{n \mid (r'_n, \eta_n) \in C\}$ is infinite. But the existence of such a set C contradicts (4). So (5) holds.

Case 2

$$(6)\qquad \begin{cases} \text{There is } s \in \omega \text{ with the following property:} \\ \text{For every } n \in \omega \text{ there are } r \in R \text{ and } \eta \in T \\ \text{such that } l(\eta) > n \text{ and } \operatorname{supp}(h(ra^s_\eta)) \not\subseteq \{\eta\}. \end{cases}$$

Fix s according to (6) and choose for every $n \in \omega$ elements $r_n \in R$ and $\eta_n \in T$ such that

$$\operatorname{supp}(h(r_n a^s_{\eta_n})) \not\subseteq \{\eta_n\} \quad \text{and} \quad l(\eta_{n+1}) > l(\eta_n).$$

Put $b_n = r_n a^s_{\eta_n}$.

We shall use the b_n to compose elements c suitable for Lemma 2.6. But first of all we have to refine our tools.

For a subset u of T put

$$\underline{l}(u) = \min\{l(\theta) \mid \theta \in u\},$$

$$\bar{l}(u) = \sup\{l(\theta) \mid \theta \in u\},$$

$$\boldsymbol{b}_\omega(u) = \min\{\boldsymbol{b}(u \cap T_{\geqq n}) \mid n \in \omega\}.$$

For $a \in \hat{G}$ put $\bar{l}(a) = \bar{l}(\operatorname{supp}(a))$, and similarly for $\underline{l}$ and $\boldsymbol{b}_\omega$.

Call an integer n a *gap level* of $a \in \hat{G}$ if $\operatorname{supp}(a) \cap T_n = \varnothing$.

By an *antichain* in T we understand a set of elements of T which are pairwise incomparable w.r.t. the order of T. An antichain of $a \in \hat{G}$ is an antichain of $\operatorname{supp}(a)$.

Gap levels and antichains are very useful in proving that an element does not belong to $G(\alpha^*)$. Indeed $a \in \hat{G}$ is not an element of $G(\alpha^*)$ if one of the following conditions is satisfied:

(a) *Gap condition.* a has infinitely many gap levels, and $\operatorname{supp}(a)$ is infinite.

(b) *Antichain condition.* a has an antichain A such that $\boldsymbol{b}(A) = \boldsymbol{b}_\omega(a)$, and $\boldsymbol{b}_\omega(a) > 0$.

This is seen by contraposition taking the second standard representation

(Proposition 2.3(3)) for a. This representation tells that there are an ordinal $\gamma < \boldsymbol{b}_\omega(a)$ and finitely many branches $w_1, \ldots, w_m$ of T such that

$$w_1 \cup \cdots \cup w_m =^* \operatorname{supp} a \setminus ({}^{\omega>}\gamma).$$

It follows easily that condition (a) is sufficient, and that $\operatorname{supp} a \setminus ({}^{\omega>}\gamma)$ cannot contain an infinite antichain. Since $\boldsymbol{b}_\omega(a) > 0$ implies that $\boldsymbol{b}_\omega(a)$ is a limit ordinal we conclude that $\boldsymbol{b}(A) < \boldsymbol{b}_\omega(a)$ for each antichain A of a, which proves the sufficiency of (b).

Now we start the discussion of case 2.

Subcase 2.1. $\operatorname{supp}(h(b_n))$ is finite for infinitely many n. Clearly $\underline{l}(h(b_n)) \geqq l(\eta_n) - s$ for all n. So we find an infinite sequence $(n_k)_{k\in\omega}$ such that

$$\min((l(\eta_{n_{k+1}})), \underline{l}(h(b_{n_{k+1}}))) > \bar{l}(h(b_{n_k})) + l(\eta_{n_k}). \tag{7}$$

Put $c = \Sigma_{k<\omega} b_{n_k}$, and pick $t \in \omega$ and $r \in R$. As

$$\operatorname{supp}(h(c) - rc^t) \supseteq \bigcup_{k<\omega} (\operatorname{supp}(h(b_{n_k})) - \{\eta_{n_k}\})$$

by (7), it follows from the choice of the b_n and η_n that the set $\operatorname{supp}(h(c) - rc^t)$ is infinite.

Equally from (7) we see that $\operatorname{supp}(h(c) - rc^t)$ has infinitely many gap levels (at least all integers $\bar{l}(h(b_{n_k})) + 1$). So the gap condition is satisfied for all elements $hc - rc^t$ $(r \in R,\ t \in \omega)$. Consequently c is as required in Lemma 2.6.

Subcase 2.2. $\operatorname{supp}(h(b_n))$ is infinite for all but finitely many n. Replacing $(b_n)_{n\in\omega}$ by a suitable subsequence we can assume that $\operatorname{supp}(h(b_n))$ is infinite for all n.

Moreover we can suppose that $h(b_n) \in G(\alpha^*)$ for all n. Indeed, if $h(b_n) \not\in G(\alpha^*)$ then $h(b_n) - rb_n^t \not\in G(\alpha^*)$ for all $r \in R$ and $t \in \omega$ since $rb_n^t \in G \subseteq G(\alpha^*)$, so $c = b_n$ proves Lemma 2.6.

So for every n we can take the second standard representation for $h(b_n)$:

$$hb_n = r_1^n a_{w_{n,1}}^s + \cdots + r_m^n a_{w_{n,m}}^s + b_n' + c_n.$$

We may assume that the sequence $(\boldsymbol{b}_\omega(h(b_n))_{n<\omega})$ is nondecreasing. (If not, replace $(b_n)_{n<\omega}$ by a suitable subsequence.)

Basic construction. Let $(\theta_n)_{n<\omega}$ be a sequence with the following properties:

(i) $\theta_n \in \operatorname{supp}(hb_n)$,

(ii) $\theta_n \not\in \{\eta_m \mid m < \omega\}$,

(iii) $\sup\{\boldsymbol{b}(\theta_n) \mid n < \omega\} = \sup\{\boldsymbol{b}_\omega(h(b_n)) \mid n < \omega\} := \beta_\omega$.

Since the sequences $l(\eta_n)$ and $\underline{l}(h(b_n))$ are unbounded, we can assume w.l.o.g. (replacing $(b_n)_{n<\omega}$ by a subsequence if necessary) that

$$(8)\qquad \begin{cases} l(\eta_{n+1}) > l(\theta_n), \\ \underline{l}(h(b_{n+1})) > \bar{l}(c_n) + l(\theta_n). \end{cases}$$

The last inequality implies $\underline{l}(c_{n+1}) > \bar{l}(c_n) + l(\theta_n)$. In fact we have $\mathrm{supp}(c_n) \subseteq \mathrm{supp}(h(b_n))$ for all n, due to the definition of c_n in the second standard representation.

Another consequence of (8) is that $\theta_n \notin \mathrm{supp}(h(b_m))$ for $n < m$.

Put $c = \Sigma_{n<\omega} b_n$.

CLAIM 1. *Assume that* $\theta_n \in \mathrm{supp}(h(c))$ *for every* $n < \omega$. *If* $h(c) - rc^t \in G(\alpha^*)$ *then* $\boldsymbol{b}_\omega(h(c) - rc^t) = \beta_\omega$, *for all* $r \in R$, $t \in \omega$.

PROOF. By property (ii) of the sequence $(\theta_n)_{n<\omega}$ every θ_n is in the support of $h(c) - rc^t$. So we have $\boldsymbol{b}_\omega(h(c) - rc^t) \geqq \beta_\omega$. Assume for contradiction that $\boldsymbol{b}_\omega(h(c) - rc^t) > \beta_\omega$ holds. Then we can choose an ordinal $\gamma > \beta_\omega$ and a nonempty collection of branches $w_1, \dots, w_k$ of T such that $\mathrm{supp}(h(c) - rc^t) \setminus ({}^{\omega >}\gamma) =^* w_1 \cup \cdots \cup w_k$. On the other hand, $\gamma > \beta_\omega$ implies

$$\mathrm{supp}(h(c) - rc^t) \setminus {}^{\omega >}\gamma \subseteq^* \bigcup \{\mathrm{supp}(c_n) \mid n \in \omega\} \cup \{\eta_n \mid n \in \omega\},$$

as clearly $\boldsymbol{b}_\omega(h(b_n)) = \boldsymbol{b}(h(b_n) - c_n)$. Consequently

$$w_1 \cup \cdots \cup w_k \subseteq^* \bigcup \{\mathrm{supp}\, c_n \mid n \in \omega\} \cup \{\eta_n \mid n \in \omega\}.$$

This is impossible since the left side has no gap levels while, by (8), the right side has infinitely many gap levels. In order to settle subcase 2.2 we consider two possibilities.

Possibility 2.2.1. There is an infinite set $\{v_n \mid n \in \omega\}$ of branches such that $v_n \subseteq^* \mathrm{supp}(h(b_n))$ for all n, and $\boldsymbol{b}(v_n) = \boldsymbol{b}_\omega(h(b_n))$.

Choosing a suitable subsequence of $(b_n)_{n\in\omega}$ we can assume that there is an antichain $(\theta_n)_{n\in\omega}$ such that $\theta_n \in v_n$ for all $n < \omega$, $\theta_n \neq \theta_m$ for $n \neq m$, and conditions (i), (ii), (iii) of the basic construction are satisfied.

Choosing a suitable subsequence of $(b_n)_{n<\omega}$ we can assume that $v_n \cap \mathrm{supp}(h(b_k))$ is finite whenever $k < n < \omega$. (Remember that $h(b_k) \in G(\alpha^*)$.) Then we can also assume that there is an antichain $(\theta_n)_{n<\omega}$ such that $\theta_n \in v_n \setminus \bigcup \{\mathrm{supp}(h(b_k)) \mid k < n\}$ for all $n < \omega$, and that conditions (i), (ii), (iii) of the basic construction are satisfied. Finally we can assume that the equations (8) hold.

As a consequence of (8) we get $\theta_n \notin \mathrm{supp}(\Sigma_{m>n} h(b_m))$ for all $n < \omega$. By (i) we have $\theta_n \in \mathrm{supp}(h(b_n))$, and $\theta_n \in v_n \setminus \mathrm{supp}(h(b_k))$ for $k < n$ implies $\theta_n \notin \Sigma_{m<n} h(b_m)$. Together this yields $\theta_n \in \mathrm{supp}(h(c))$ for all $n < \omega$. It follows that $(\theta_n)_{n<\omega}$ is an antichain of $h(c) - rc^t$ for all $r \in R$ and $t < \omega$. Claim 1 can be applied. Thus if $h(c) - rc^t \in G(\alpha^*)$ then the antichain condition is satisfied for $h(c) - rc^t$. We conclude that $hc \notin SG(G(\alpha^*) \cup \{c^t \mid t < \omega\})$. So in this situation the lemma holds true.

Possibility 2.2.2. There is a finite set of branches $\{w_1, \dots, w_m\}$ such that for every n and every branch $v \subseteq^* \mathrm{supp}(h(b_n))$ with $\boldsymbol{b}(v) = \boldsymbol{b}_\omega(h(b_n))$ we have $v \in \{w_1, \dots, w_m\}$. In this case we have for every n the representation

$$(9)\qquad \begin{cases} h(b_n) = r_1^n a_{w_1}^{s,n} + \cdots + r_m^n a_{w_m}^{s,n} + b'_n + c_n \\ \text{with} \\ a_{w_k}^{s,n} = \displaystyle\sum_{\substack{\eta \in w_k \\ l(\eta) \geqq s \\ l(\eta) \geqq l(b_n)}} (p^{l(\eta)-s}) \cdot \eta \end{cases}$$

and $\boldsymbol{b}(w_1) = \cdots = \boldsymbol{b}(w_m) > \boldsymbol{b}(b'_n)$, $c_n \in G$, $\boldsymbol{b}(\theta) > \boldsymbol{b}(w_1)$ for all $\theta \in \mathrm{supp}(c)$.

We can assume that $r_k^n a_{w_k}^{s,n} \neq 0$ for all $k \leqq m$. (If this is not the case we can satisfy this condition for an infinite subsequence of $(b_n)_{n<\omega}$ and a nonempty subset of $\{w_1, \dots, w_m\}$ using the pidgeon-hole principle.)

Since the property of Possibility 2.2.2 is inherited by subsequences of $(b_n)_{n<\omega}$ we can assume that (8) holds, and so we can construct c using the basic construction with an arbitrary sequence $(\theta_n)_{n<\omega}$ satisfying (i), (ii), (iii).

Claim 2. *If in* (9) *we have* $c_n = 0$ *for all* n, *then* $h(c) - rc^t \notin G(\alpha^*)$ *for all* $r \in R$ *and* $t \in \omega$.

Before embarking on the proof we show how Possibility 2.2.2 reduces to Claim 2. Assume that (9) holds, but $c_n \neq 0$ for some n, and that $h(c) - r'c^{t'} \in G(\alpha^*)$ for some $r' \in R$ and $t' \in \omega$. Using $\beta_\omega = \boldsymbol{b}(w_1) < \boldsymbol{b}(c_n)$ for all n and taking subsequences we can concentrate on the case $c_n = r'b_n^{t'}$ for all n.

For every subset $A \subseteq T$ the mapping π_A, which assigns to each element $\Sigma_{\eta \in w} r_\eta \cdot \eta$ of $\hat{G}$ the element $\Sigma_{\eta \in w \cap A} r_\eta \cdot \eta$, is a homomorphism.

Put $u = \mathrm{supp}(h(c) - r'c^{t'})$. Combining (8) and (9) we see that $\mathrm{supp}(c_n) \cap u = \varnothing$ for all $n \in \omega$. So clearly (9) holds for $\pi_u \cdot h$ instead of h with $c_n = 0$ for all n. As $\mathrm{supp}(h(c)) \,\triangle\, \mathrm{supp}(\pi_u h(c)) = \{\eta_n \mid n < \omega\}$, the sequence $(\theta_n)_{n<\omega}$ and the element c satisfy properties (i), (ii), (iii) and (8) for h_u, too.

So we can apply Claim 2 to $\pi_u h$ and get $\pi_u h(c) \notin G(\alpha^*)$ setting $r = 0$. But

$\pi_u h(c) = h(c) - r'c^{t'}$, so $\pi_u hc \in G(\alpha^*)$ by assumption, a contradiction. So in fact Possibility 2.2.2 reduces to Claim 2.

A first step in the proof of Claim 2 is the following

CLAIM 3. *Let h and c be as in Claim 2. Assume $h(c) - rc^t \in G(\alpha^*)$. Let U be the set of infinite maximal chains $u \subseteq \operatorname{supp}(h(c)) \setminus (w_1 \cup \cdots \cup w_m)$ such that $\boldsymbol{b}(u) = \boldsymbol{b}(w_1)$.*

Let $B = \operatorname{supp}(h(c)) \setminus (\bigcup U \cup w_1 \cup \cdots \cup w_m)$. Then replacing $(b_n)_{n<\omega}$ by a suitable subsequence we obtain

(a) $U = \emptyset$,

(b) $\boldsymbol{b}(B) < \boldsymbol{b}(w_1)$.

PROOF OF CLAIM 3. If U is infinite and no element $u \in U$ satisfies $u \subseteq^* \{\eta_n \mid n < \omega\}$ then $\bigcup U$ contains an infinite antichain A with $A \cap \{\eta_n \mid n < \omega\} = \emptyset$. So $A \subseteq \operatorname{supp}(h(c) - rc^t)$, and the antichain condition yields a contradiction. Thus we can assume that there is an element $u \in U$ such that $u \subseteq^* \{\eta_n \mid n < \omega\}$. Choosing a suitable subsequence of $(b_n)_{n<\omega}$ we can assume that $v \cap \{\eta_n \mid n < \omega\}$ is finite for all $v \in U$ with $v \neq u$. So we find an antichain $A \subseteq \bigcup U$ such that $|A| = |U| - 1$ and $A \cap \{\eta_n \mid n < \omega\} = \emptyset$. The antichain condition implies that U is finite, say $U = \{u_0, \ldots, u_j\}$.

For $k \leqq j$ we have $\boldsymbol{b}(u_k) = \boldsymbol{b}(w_1) > \boldsymbol{b}(b_n)$. So we find $\bar{n} < \omega$ such that $\bar{l}(u_k \cap \operatorname{supp}(b'_n)) \leqq \bar{n}$ for all $k \leqq j$. Due to the definition of $a^{\varsigma, n}_{w_k}$ in (9) the sequence $(\underline{l}(b'_n))_{n<\omega}$ is unbounded, so we can assume that $\underline{l}(b_{n+1}) > \bar{n} + 2$ for all $n < \omega$. Assume that there exists $v \in U \setminus \{u\}$, and let w be the branch of T containing v. It follows that $\operatorname{supp}(h(c) - rc^t) \cap w$ is infinite, has breadth $\boldsymbol{b}(w_1)$ and infinitely many gap levels (we can assume that η_n, η_m never belong to consecutive levels). On the other hand the second standard representation implies $\operatorname{supp}(h(c) - rc^t) \cap w =^* v_\alpha$ for some branch v_α, a contradiction.

Thus we can assume that $U = \emptyset$ or $U = \{u\}$. If $U = \{u\}$ then $\{\eta_n \mid n < \omega\} \cap \operatorname{supp}(h(c) - rc^t)$ is finite since $\{\eta_n \mid n < \omega\}$ is a chain with breadth $\boldsymbol{b}(w_1)$ and infinitely many gap levels. We can assume $\{\eta_n \mid n < \omega\} \cap \operatorname{supp}(h(c) - rc^t) = \emptyset$. Hence with $A = T \setminus \{\eta_n \mid n < \omega\}$ we obtain $\pi_A h(c) = h(c) - rc^t$, and for $\pi_A h$ in the role of h we have $U = \emptyset$. Moreover $\pi_A h$ satisfies the conditions of Claim 2. So once we have shown Claim 2 with the additional assumption $U = \emptyset$ we can deduce $\pi_A h(c) - r'c^{t'} \notin G(\alpha^*)$ for all $r' \in R$, $t' \in \omega$, and we get the desired result for h putting $r' = 0$.

Now assume $U = \emptyset$, and suppose that $\boldsymbol{b}(B) = \boldsymbol{b}(w_1)$. As $U = \emptyset$ there is an antichain A in B with $\boldsymbol{b}(A) = \boldsymbol{b}(w_1)$, contradicting the antichain condition. This completes the proof of Claim 3.

Claim 2 is proved indirectly. So assume $h(c) - rc' \in G(\alpha^*)$. Using Claim 3 and the fact that R^+ is reduced in connection with (9) one easily gets a sequence $(\theta_n)_{n<\omega}$ of elements of w_1 satisfying $\theta_n \in \mathrm{supp}(h(c) - rc')$ for all n. So the second standard representation of $h(c) - rc'$ looks as follows:

$$h(c) - rc' = r_1 a^s_{w_1} + \cdots + r_m a^s_{w_m} + b + d, \qquad r_1 a^s_{w_1} \neq 0. \tag{10}$$

Let w be the set of all $\theta \in \mathrm{supp}(h(c) - rc')$ with $\theta \in w_1 \setminus (w_2 \cup \cdots \cup w_m \cup \mathrm{supp}(b + d))$. Clearly w contains an end segment of the branch w_1. Therefore we find n_0 as well as $\theta \in w$ such that $l(\theta) = \underline{l}(h(b_{n_0})) - 1$ and $\theta' \in w$ for all $\theta' \in w_1$ with $l(\theta') > l(\theta)$. Pick $\theta' \in w$ with $l(\theta') = \underline{l}(h(b_{n_0}))$. Comparing the summands containing θ resp θ' in (10) and in the second standard form version of (9) one gets

$$r_1 a^s_\theta = \left(\sum_{n<n_0} r^n_1 \right) a^s_\theta \quad \text{and} \quad r_1 a^s_{\theta'} = \left(\sum_{n \leqq n_0} r^n_1 \right) a^s_{\theta'}$$

(check this using the conditions on θ and θ' stated above). Consequently $(r_1 - \Sigma_{n<n_0} r^n_1) \equiv 0 \bmod p^{s+1}$. Hence $r^{n_0}_1 a^s_{\theta'} = 0$ which means $r^{n_0}_1 \equiv 0 \bmod p^{s+1}$. Hence $r^{n_0}_1 a^{s,n_0}_{w_1} = 0$, contradicting (9). So Claim 2 is confirmed, and the proof of Theorem 2.1 is complete.

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